AVERAGE AND PROBABILITY.

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A random diameter is drawn in a given circle. Find the chance that it intersects, (1) a random chord; (2) a random chord through a random point; and (3) a chord through two random points.

Solution by the PROPOSER.

Let O be the center of the given circle, AB the random chord, M the one random point, and N the second random point.

Let $OA=r$, $AM=x$, $\angle AOH=\theta$. For the second point, let $MN=y$ and let μ=angle AB makes with some fixed line

Then $AH=r\sin\theta$; an element of the circle at M is $r\sin\theta\, d\theta\, dx$; at n, $d\mu\, ydy$. The limits of θ are 0 and $\frac{1}{2}\pi$; of x, 0 and $2r\sin\theta=x'$; of y, 0 and x and doubled. Then the required chance is

$$(1)\quad p=\frac{\int_0^{\frac{1}{2}\pi}4\theta\, d\theta}{\int_0^{\frac{1}{2}\pi}2\pi\, d\theta}=\frac{4}{\pi^2}\int_0^{\frac{1}{2}\pi}\theta\, d\theta=\tfrac{1}{2}.$$

$$(2)\quad p=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{x'}4r\theta\sin\theta\, d\theta\, dx}{\int_0^{\frac{1}{2}\pi}\int_0^{x'}2\pi r\sin\theta\, d\theta\, dx}=\frac{4}{\pi^2 r}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\theta\sin\theta\, d\theta\, dx$$

$$=\frac{8}{\pi^2}\int_0^{\frac{1}{2}\pi}\theta\sin^2\theta\, d\theta=\tfrac{1}{2}+\frac{2}{\pi^2}.$$

$$(3)\quad p=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_0^{x}\int_0^{2\pi}4r\theta\sin\theta\, d\theta\, dx\, d\mu\, ydy}{\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_0^{x}\int_0^{2\pi}2\pi r\sin\theta\, d\theta\, dx\, d\mu\, ydy}$$

$$=\frac{4}{\pi^3 r^3}\int_0^{\frac{1}{2}\pi}\int_0^{x'}\int_0^{x}\int_0^{2\pi}\theta y\sin\theta\, d\theta\, dx\, dy\, d\mu$$

$$=\frac{4}{\pi^2 r^3}\int_0^{\frac{1}{2}\pi}\int_0^{x'}x^2\theta\sin\theta\, d\theta\, dx=\frac{32}{3\pi^2}\int_0^{\frac{1}{2}\pi}\theta\sin^4\theta\, d\theta=\tfrac{1}{2}+\frac{8}{3\pi^2}.$$

If a random chord is drawn instead of a random diameter, we let $\angle COK=\phi$, $\angle AOC=\psi$. The limits of ϕ are 0 and θ and doubled; of ψ, 0 and 2ϕ and doubled.

$$\therefore (1)\quad p=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi} d\theta\, d\phi\, d\psi}{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{\pi} d\theta\, d\phi\, d\psi}=\frac{8}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi} d\theta\, d\phi\, d\psi$$

$$=\frac{16}{\pi^3}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\phi\, d\theta\, d\phi=\frac{8}{\pi^3}\int_0^{\frac{1}{2}\pi}\theta^2 d\theta=\tfrac{1}{3}\text{ (same as 191).}$$

$$(2)\quad p=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}\int_0^{x'} r\sin\theta\, d\theta\, d\phi\, d\psi\, dx}{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{\pi}\int_0^{x'} r\sin\theta\, d\theta\, d\phi\, d\psi\, dx}$$

$$=\frac{8}{\pi r(\pi^2+4)}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}\int_0^{x'}\sin\theta\, d\theta\, d\phi\, d\psi\, dx$$

$$=\frac{16}{\pi(\pi^2+4)}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}\sin^2\theta\, d\theta\, d\phi\, d\psi=\frac{16}{\pi(\pi^2+4)}\int_0^{\frac{1}{2}\pi}\theta^2\sin^2\theta\, d\theta$$

$$=\frac{\pi^2+6}{3(\pi^2+4)}=\tfrac{1}{3}+\frac{2}{3(\pi^2+4)}.$$

$$(3)\quad p=\frac{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}\int_0^{x'}\int_0^{x} yr\sin\theta\, d\theta\, d\phi\, d\psi\, dx\, dy}{\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{\pi}\int_0^{x'}\int_0^{x} yr\sin\theta\, d\theta\, d\phi\, d\psi\, dx\, dy}$$

$$=\frac{48}{\pi r^3(3\pi^2+16)}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}\int_0^{x'}\int_0^{x} y\sin\theta\, d\theta\, d\phi\, d\psi\, dx\, dy$$

$$=\frac{64}{\pi(3\pi^2+16)}\int_0^{\frac{1}{2}\pi}\int_0^{\theta}\int_0^{2\phi}\sin^4\theta\, d\theta\, d\phi\, d\psi=\frac{64}{\pi(3\pi^2+16)}\int_0^{\frac{1}{2}\pi}\theta^2\sin^4\theta\, d\theta$$

$$=\frac{\pi^2+9}{3\pi^2+16}=\tfrac{1}{3}+\frac{11}{3(3\pi^2+16)}.$$

This exhibits the reason for the $\frac{1}{2}$ given by some contributors to 191 instead of the correct value $\frac{1}{3}$.